International Journal of Theoretical Physics, Vol. 33, No. 3, 1994

Topology and Statistics in Zero Dimensions

Charilaos Aneziris[1]

Received March 3, 1993

It has been suggested that space-time may be intrinsically not continuous, but discrete. Here we review some topological notions of discrete manifolds, in particular ones made out of finite number of points, and discuss the possibilities for statistics in such spaces.

1. INTRODUCTION

One of the most basic assumptions of disciplines such as classical and quantum mechanics and general relativity is that space-time is continuous, and therefore it makes sense to consider derivatives with respect to space or time coordinates. This assumption, however, has been challenged recently by developments in quantum gravity, among others, where space-time may be considered continuous only over large lengths with respect to the Planck scale, because it is essentially discrete. In order to avoid, for instance, infinite black hole entropy, one has to define a minimum length, that is, a minimum distance between two discrete points. Subsequently, the volume of a finite region of space-time is proportional to the number of points in this region, and this number is thus finite.

In a recent paper, Sorkin (1991) discussed possible topologies for such spaces, and showed that even though they are discrete, they may possess nontrivial homotopy groups. In Section 2 here, we briefly review some of his work, in order to find some general rules in Section 3. In Section 4 we discuss the problem of statistics in finite spaces, while in Section 5 we conclude by mentioning some open questions.

[1]Physics Department, Weizmann Institute of Science, Rehovot, 76100 Israel.

0020-7748/94/0300-0535$07.00/0

2. TOPOLOGY FOR FINITE SPACES

2.1. General Notions of Topology for Partially Ordered Sets

Let S be any set, finite or infinite. We say that a topology is defined on S, once we are provided with a collection $\mathscr{T}$ of subsets of S, satisfying the following:

(i) $\emptyset, S \in \mathscr{T}$.
(ii) The union of any number of elements of $\mathscr{T}$, finite or infinite, belongs to $\mathscr{T}$.
(iii) The intersection of any *finite* number of elements of $\mathscr{T}$ belongs to $\mathscr{T}$.

The elements of such a collection $\mathscr{T}$ are called *open sets*, and once the open subsets of a set S are given, we are able to define the notion of continuity and processed to find the topological properties of S. It is therefore obvious that these properties do not depend only on the set S itself, but also on the topology defined on it.

Now let x be any element of S. Let $\Lambda(x)$ be the intersection of all open sets to which x belongs:

$$\Lambda(x) = \bigcap\{A \in \mathscr{T} : x \in A\} \tag{2.1}$$

This set is called the *smallest neighborhood of* x. For finite spaces $\Lambda(x)$ is obviously open. This need not be the case, however, for infinite spaces; for example, in a Euclidean space with the usual topology, $\Lambda(x) = \{x\}$, which is not open.

We are now going to provide S with the following partial order:

$$x \to y \Leftrightarrow x \in \Lambda(y) \tag{2.2}$$

While (2.2) guarantees that $x \to x$ and $x \to y \wedge y \to z \Rightarrow x \to z$, it is possible that one may have $x \to y$ *and* $y \to x$ for $x \neq y$. In order to avoid such a situation, we identify any such elements. In other words, the partial ordering is defined not on S, but on the coset $S/\sim$, where by $x \sim y$ we mean $x \to y \wedge y \to x$.

Conversely, once a set S admits a partial order $\to$, one can define a topology on it as follows. A set is open iff it can be written as a union of (zero, one, or more) sets $\Lambda(x) = \{y \in S : y \to x\}$.

2.2. Topological Properties of Finite Spaces

Let $F = \{x_1, x_2, \ldots, x_n, n \in N\}$ be a finite set partially ordered by some relation $\to$. By applying the technique mentioned in the end of Section 2.1 we can define a topology on F. Schematically this can be shown by a diagram such as the one shown in Fig. 1, where $x_1, x_2, \ldots, x_n$ are the

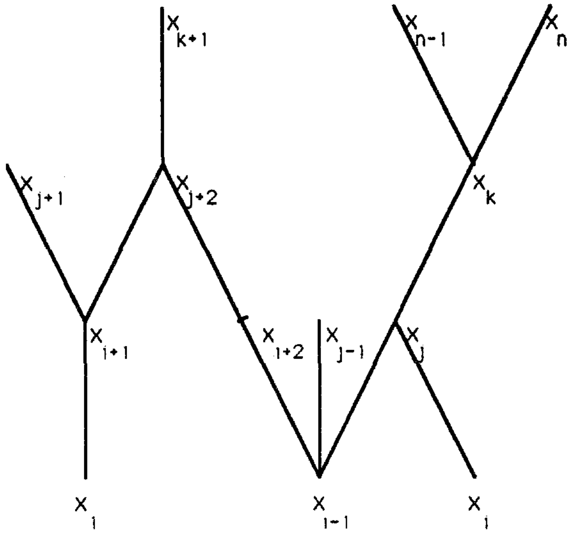

Fig. 1.

elements of F, and the partial order is the following: if x_i is directly linked to x_j (i.e., there is no x_k between them) and x_j is placed above x_i, then $x_i \to x_j$. Obviously $x_i \to x_j$ and $x_j \to x_k$ implies $x_i \to x_k$. Diagrams such as the ones shown in Fig. 1, which uniquely define a topology on a finite set, are called *Hasse* diagrams. For such a topology one may easily check that an open set consists of a number of elements $x_a, x_b, \ldots, x_l$, and all elements x_m for which $x_i \to x_m$ for at least one $i \in \{a, b, \ldots, l\}$.

Once a topology is defined on a finite space, one would be interested first to check if this space is connected, and then to find the homotopy groups. Obviously under the usual topology defined in continuous spaces a finite space is disconnected, since there is no continuous path connecting any two distinct elements, and thus all homotopy groups are trivial. This need not be the case here, however, for the topology defined in this paper. As Sorkin (1991) has shown, for a space consisting of four points whose Hasse diagram is the one shown in Fig. 2, let $f(t)$ be the following mapping from F to S^1:

$$f(x) = \begin{cases} x_3 & \text{if} \quad t \in \{0, 1\} \\ x_1 & \text{if} \quad 0 < t < \frac{1}{2} \\ x_4 & \text{if} \quad t = \frac{1}{2} \\ x_2 & \text{if} \quad \frac{1}{2} < t < 1 \end{cases} \tag{2.3}$$

One can check that for any open set $u \subseteq S^1$, $f^{-1}(u)$ is open in F, and

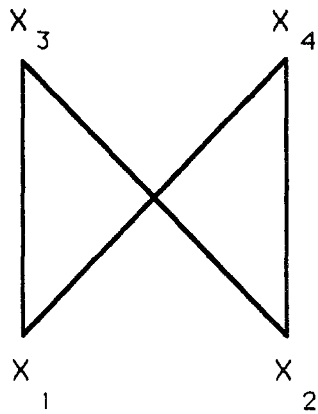

Fig. 2.

therefore $f(t)$ is a continuous function, and can therefore be considered as a loop in F. This loop, however, cannot be continuously deformed to identity, and thus the fundamental group $\pi_1(F)$ is not trivial. We will show later that in fact it is Z.

In Section 3 we continue this discussion in order to find the general rules for the connectedness and the first homotopy (fundamental) group of a finite set.

3. CONTINUITY, CONNECTEDNESS, AND HOMOTOPY GROUPS

3.1. Continuity

Let f be a function mapping a set S_1 to a set S_2. The function f is called continuous iff its inverse, f^{-1}, maps every open subset of S_2 to an open subset of S_1. In particular, if $S_2 = \{x_1, x_2, \ldots, x_n\}$ is a finite partially ordered set with a topology described on Section 2, this amounts to the following.

Let $A_i \subseteq S_1$ be the set $\{s \in S_1 : f(s) = x_i\}$, and $\Lambda(x_i) = \{x_j \in S_2 : x_j \to x_i\} = \{x_a, x_b, \ldots, x_i\}$. Then a function f is continuous iff $A_a \cup A_b \cup \cdots \cup A_i$ is open $\forall x_i \in S_2$.

3.2. Connectedness

Let now $S_1 = [0, 1] = \{x \in r : 0 \leq x \leq 1\}$ having the usual topology, and f a function mapping S_1 to a finite set F. We are interested in finding a criterion that will enable us to find whether f is continuous, and once this is known, whether F is connected, and if not which are its connected components.

A function f mapping [0, 1] to F may be written in general as follows:

$$f(t) = \begin{cases} x_{i_1} & \text{if} \quad 0 < t < t_1 \\ x_{i_2} & \text{if} \quad t_1 < t < t_2 \\ \dots & \dots \\ x_{i_{r-1}} & \text{if} \quad t_{r-2} < t < t_{r-1} \\ x_{i_r} & \text{if} \quad t_{r-1} < t < 1 \end{cases} \tag{3.1}$$

while $f(0) = x_{j_0}, f(1) = x_{j_r}$, and $f(t_k) = x_{j_k}$, where $1 \leq k \leq r - 1$. The points $x_{i_1}, \dots, x_{i_r}, x_{j_0}, \dots, x_{j_r}$ need not be distinct, and r need not be finite.

Let now $x_a \in F$ be an element at the "bottom" of the Hasse diagram, which means that there is no $i \neq a$ such that $x_i \to x_a$. According to what we have already shown, $\{x_a\}$ is an open set. Then in order for f to be continuous, $A_a = f^{-1}(x_a)$ must be open, and thus $x_a \neq x_{j_k}$ unless $x_{i_{k+1}} = x_{i_k} = x_a$, that is, unless $x_{i_{k+1}}, x_{i_k} \to x_{j_k}$ for $x_{j_k} = x_a$.

Let now x_b be the element "immediately above" x_a, that is, $x_a \to x_b$ and there is no $i \neq a, b$ such that $x_i \to x_b$. Now $\{x_b\}$ is not open, but $\{x_a, x_b\}$ is, and thus if f is continuous, $A_a \cup A_b$ is open. Therefore x_b can be equal to x_{j_k} only if $x_{i_{k+1}}$ and x_{i_k} are either x_a or x_b, in which case $x_{i_{k+1}}, x_{i_k} \to x_{j_k}$. Similarly, one can show by induction that if f is continuous, then $\forall n$: $x_{i_{k+1}}, x_{i_k} \to x_{j_n}$.

Therefore x_m and x_n belong to the same connected component of F iff it is possible to find a series of points $x_p, x_q, \dots, x_r$ such that $x_m \leftrightarrow x_p \leftrightarrow x_q \leftrightarrow \dots \leftrightarrow x_r \leftrightarrow x_n$, where here by $x \leftrightarrow y$ we mean $x \to y \vee y \to x$.[2] Schematically, two elements belong to the same connected component if there is a continuous line on the Hasse diagram, not necessarily straight, linking these two points. Therefore, if the Hasse diagram is connected, the corresponding partially ordered set is connected, too; if not, each connected component of the Hasse diagram corresponds to a connected component of the set (see Fig. 3).

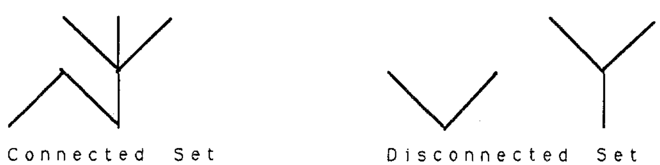

Fig. 3.

[2]There is an alternative proof of this statement in Stong (1966).

3.3. Fundamental Groups

Let l be a function mapping S^1 to F. If l is continuous, it is called a *loop*, and the point $l(0) = l(1)$ is called the base point.

Two loops l and l' that have the same base point are called *homotopic* and belong to the same (first) homotopy class iff they can be continuously deformed to each other. It is well known that the set of homotopy classes forms a group, which is called the *fundamental* group.

Now let l be a loop whose base point is x_{j_0}. According to what we said in Sections 3.1 and 3.2, if l is the loop shown in Fig. 4, the following relations should be valid:

$$x_{i_1}, x_{i_2} \to x_{j_1}, \qquad x_{i_2}, x_{i_3} \to x_{j_2}, \ldots, \qquad x_{i_r}, x_{i_1} \to x_{j_0} \tag{3.2}$$

In order for l to be trivial, one must be able to continuously deform it to the constant loop $f(t) = x_{j_0}$ $\forall t \in S^1$. One might attempt such a deformation as follows.

First one could "expand" the area "occupied" by $x_{j_0}, \ldots, x_{j_r - 1}$ from a single point to a larger and larger arc, at the expense of their "neighbors" $x_{i_1}, \ldots, x_{i_r}$. When the latter are "squeezed out," one might go on to eliminate the points $x_{j_1}, \ldots, x_{j_r - 1}$, until finally the whole circle "belongs" to x_{j_0}.

Such a deformation, however, is not always continuous. The first step, for example, is not continuous unless either $x_{j_s - 1} \to x_{j_s}$ or $x_{j_s} \to x_{j_s - 1}$. Similarly the second step may not be permitted, unless appropriate relations among the x_j are satisfied.

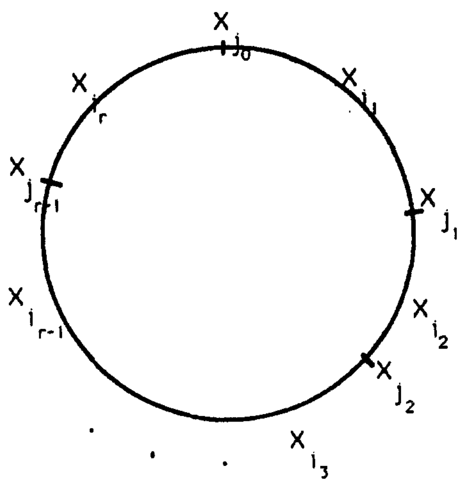

Fig. 4.

The general rule could be stated as follows. Let x_a "occupy" an area between x_a and x_c, as shown in Fig. 5. One can continuously increase the areas "occupied" by x_a and x_c at the expense of x_b. In order, however, to be able to "throw away" x_b through a continuous deformation, one must have either $x_a \rightarrow x_c$ or $x_c \rightarrow x_a$. Since the loop we began with is a continuous

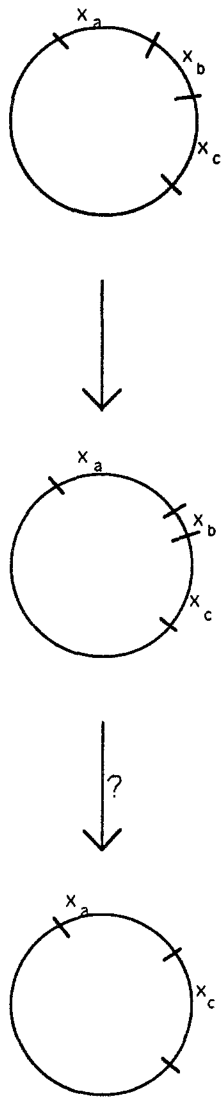

Fig. 5.

function, we already know that $x_a \to x_b \vee x_b \to x_a$, and $x_b \to x_c \vee x_c \to x_b$. Therefore the "elimination" of x_b is possible only if the triad of (x_a, x_b, x_c) is totally ordered.

Therefore a loop is trivial iff it can be deformed to the constant loop by a series of steps like the ones described above. It may be interesting to notice that this is equivalent to saying that a loop is trivial iff the corresponding loop in the Hasse diagram is trivial, where, however, we have to assume that diagonal links do not "touch" and totally ordered triads are considered "aligned." Therefore a partially ordered finite set has the same fundamental group with a corresponding Hasse diagram.

Let us discuss now some specific examples, which are illustrated in Fig. 6. First, let the finite set $F = \{x_1\}$ contain just one point. It is obvious that the only possible loop is the constant one, and thus the fundamental group is trivial.

For $F = \{x_1, x_2\}$ there are two possible partial orderings:

(a) The one for which x_1 and x_2 are not related, and thus F is disconnected. In this case the only possible loops are the two constant ones $f(t) = x_1$ and $f(t) = x_2$.

(b) The one for which $x_1 \to x_2$. In addition to the constant loops, one may also have loops such as the one shown in Fig. 7, where all "boundary" points are "occupied" by x_2. One can easily see that all such loops are trivial. (The case $x_2 \to x_1$ yields, of course, exactly the same results.)

For $F = \{x_1, x_2, x_3\}$ it is possible to have two or three distinct connected components, but such cases can be reduced to the ones discussed

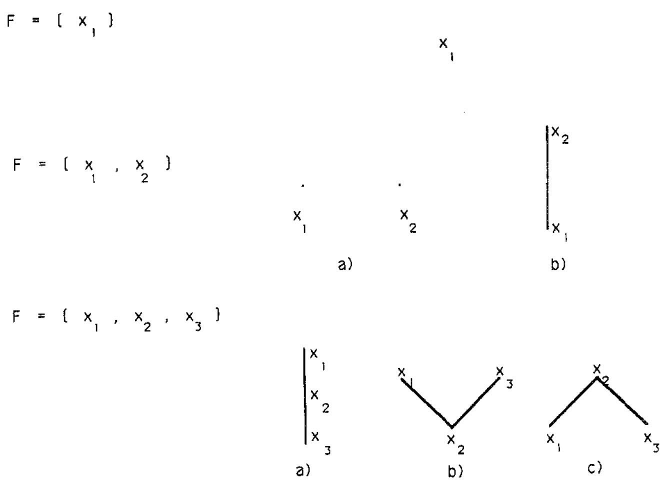

Fig. 6.

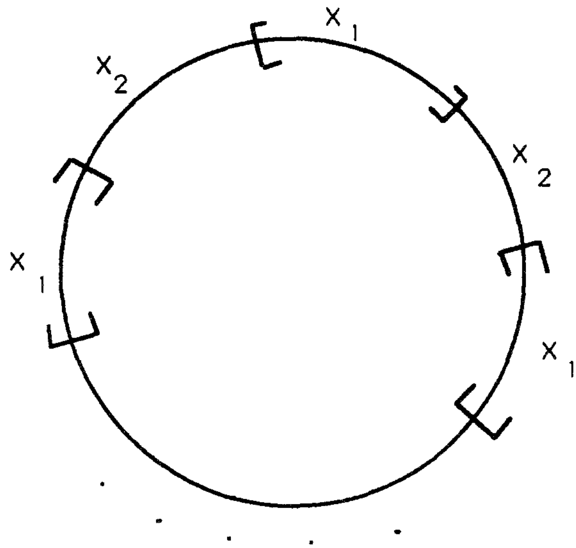

Fig. 7.

above. We are thus going to focus on the case where F is connected, for which there are essentially three possibilities:

(a) $x_1 \rightarrow x_2 \rightarrow x_3$ (total ordering).
(b) $x_2 \rightarrow x_1$ and $x_2 \rightarrow x_3$.
(c) $x_1 \rightarrow x_2$ and $x_3 \rightarrow x_2$.

The fundamental group for (a) is obviously trivial. For (b) and (c) one may continuously eliminate the areas "occupied" by x_1 and x_3, until the whole circle "belongs" to x_2, as shown in Fig. 8. (We have assumed above that x_2 is the base point; this is irrelevant, however, since the fundamental group does not depend on the base point.)

Therefore for F containing one, two, or three points all loops are trivial and thus $\pi_1(F) = e$.

Let now $F = \{x_1, x_2, x_3, x_4\}$ contain four points and be connected. Then there are ten possible orderings which are shown in Fig. 9. One can see that the first nine possibilities allow for only trivial loops; the last one, however, which we already mentioned in Section 2 and has been discussed by Sorkin (1991), allows for the nontrivial loop given by (2.3). According to the rules we already gave in this subsection, not only this loop, but all (nonzero) powers of this loop are nontrivial, and thus $\pi_1(F) = Z$. This is not surprising, since the Hasse diagram in this case is a circle (we recall that the diagonals do not "touch" each other) and it is well known that the

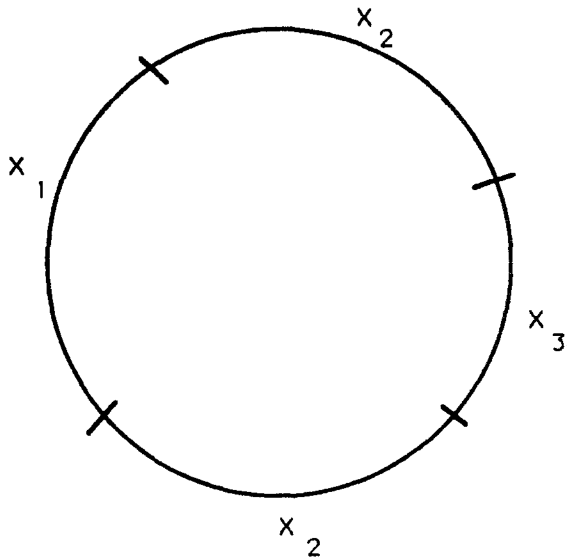

Fig. 8.

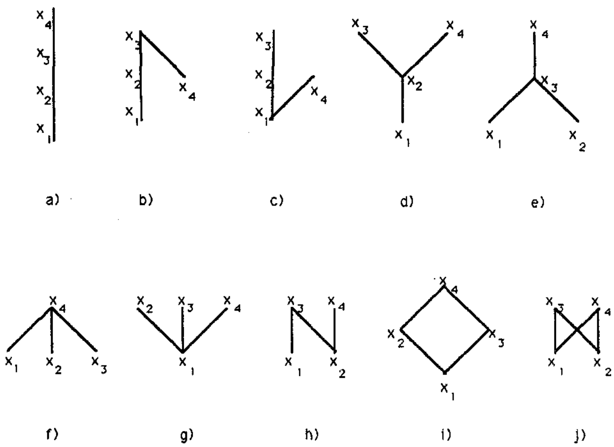

Fig. 9.

fundamental group of a circle is Z. What may seem surprising, in fact, is our statement that the fundamental group of the ninth case is trivial instead of Z, since from a first look, its Hasse diagram looks like a circle, too; one should notice, however, that even though, for the sake of clarity, we have not drawn the four points on the same line, indeed they should be considered collinear since the triads (x, x_2, x_4) and (x_1, x_3, x_4) are totally ordered.

One may similarly proceed for spaces containing five or more points; although the number of possible topologies increases very rapidly and in each case one has to be very careful to avoid overcounting or undercounting, the same rules apply and in principle it is always possible to find the fundamental group.

3.4. Higher Homotopy Groups

Having discussed the fundamental group of finite sets, one may proceed along the same lines to see what happens for the second homotopy group. Unfortunately we were not able to find a simple rule like the one we gave for the fundamental group; we were only able to show that one needs a minimum of six points ordered as shown in Fig. 10 in order to have a nontrivial mapping from S^2 to F. The generator of the second homotopy group Z of Fig. 10 is one mapping each of the six points to a face of a die, i.e., x_i and x_{7-i} are mapped to opposite faces. An edge joining faces x_i and x_j, where $x_i \to x_j$, is mapped to x_j, and a vertex joining x_i, x_j, and x_k, where $x_i \to x_j \to x_k$, is mapped to x_k.

One can continue the same way for higher homotopy groups; in accordance with Sorkin (1991), one needs a minimum of $2n+2$ points ordered as in Fig. 11 in order to get a nontrivial mapping from S^n to F, and thus in order to have nontrivial $\pi_n(F)$.

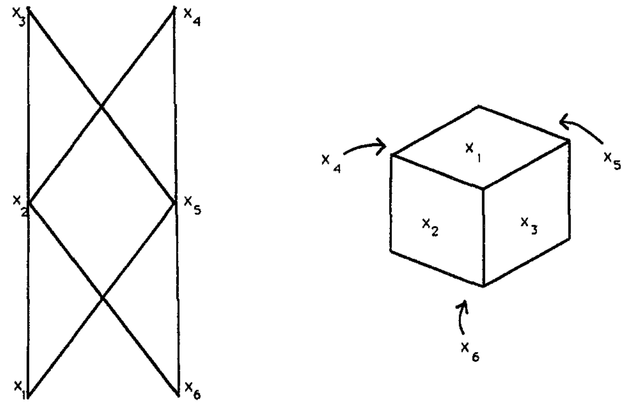

Fig. 10.

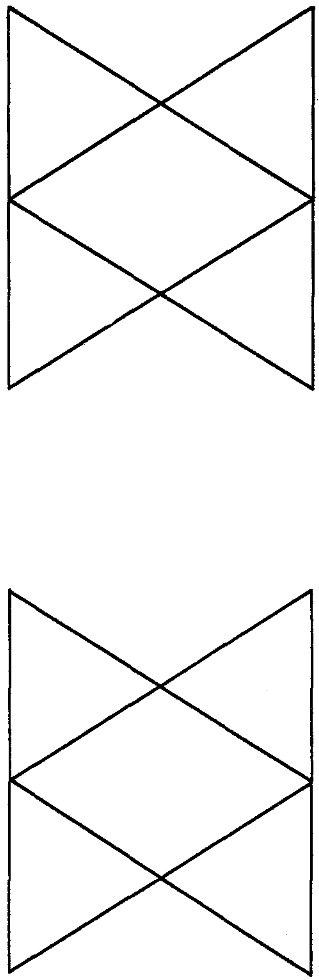

Fig. 11.

4. STATISTICS IN FINITE SPACES

4.1. A Brief Review on Statistics

While the study of finite spaces is an interesting subject on its own merit, it is more relevant to physics to discuss what happens when actual particles are placed on such manifolds. In this section we focus on the possibilities for statistics which arise when two or more identical particles are placed in a finite space.

Let F be a finite space on which we place two identical particles. As is well known from quantum mechanics, identical particles are considered indistinguishable. In addition, following an assumption valid for continuous spaces, one is not allowed to place two distinct particles on the same point of F. Therefore while the physical space is F, the configuration space is $(F \times F \backslash F)/Z_2$, that is,

$$Q = \{(x_i, x_j): x_i, x_j \in F, x_i \neq x_j, (x_i, x_j) \equiv (x_j, x_i)\} \tag{4.1}$$

It is also known that while the observables are functions on the configuration space Q, the wave function is not on Q, but on its universal cover $\bar{Q}$. It has been shown that this implies the existence of n possibilities for quantization for each n-dimensional representation of $\pi_1(Q)$. This was in fact our motivation in studying the fundamental groups of finite spaces in Section 3, and of the configuration space in this section.

In addition, once a particular representation is chosen, it maps the exchange, which is of course a loop on Q, to some unitary matrix. This matrix characterizes the statistics of a physical system, and thus the knowledge of $\pi_1(Q)$ enables us to find the possibilities for statistics.[3]

4.2. Statistics for a Few Simple Cases

Since the two identical particles are not allowed to be on the same point, there is no need to consider the case $F = \{x_1\}$.

Let $F = \{x_1, x_2\}$. The configuration space of F consists of just one point, where one particle is on x_1 and the other on x_2. Therefore the only possible loop is the constant one, and this is true for both possible topologies of F.

While it may not be relevant for the problem of statistics, one might ask what would happen if the two particles were distinguishable. In such a case one might consider an exchange process where the first particle would be located at x_1 for $0 \leq t < t_0$ and at x_2 for $t_0 < t \leq 1$, and the second conversely. Such a process is not a loop, of course, in the configuration space of *distinguishable* particles. In fact, it is not even a (continuous) path, since in order for the trajectories of both particles to be continuous, one requires both $x_1 \rightarrow x_2$ and $x_2 \rightarrow x_1$, which, as mentioned in Section 2, is forbidden.

We now move on to $F = \{x_1, x_2, x_3\}$. It is fairly obvious that unless we consider a total ordering $x_1 \rightarrow x_2 \rightarrow x_3$, no exchange is allowed and thus the question of statistics is meaningless. When $x_1 \rightarrow x_2 \rightarrow x_3$, however, one may perform exchanges such as the ones shown in Fig. 12, where the trajectory

[3]A detailed review of quantum statistics can be found, e.g., in Balachandran (1991) or Sorkin (1986).

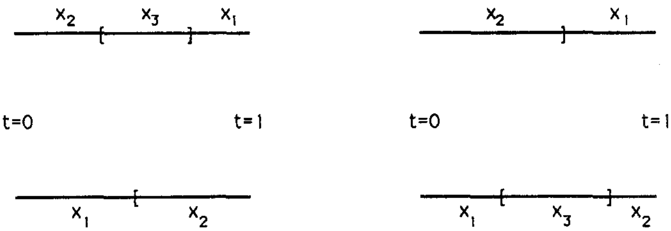

Fig. 12.

of the first particle is shown at the top and that of the second particle at the bottom (we will use this convention for the next figures as well). These loops are of course nontrivial, since the trajectory of each particle is not closed and thus cannot be shrunk to identity. In Fig. 13 we show the product of these two loops, which can be shown to be trivial as follows. First one eliminates x_1 from the trajectory of the first particle, then x_2 from the trajectory of the second particle, and finally x_3 from the trajectory of the first. In Fig. 14 we show the square of one of the loops, and by exhausting all possible deformations one can see that none of the intermediate segments may be eliminated without destroying the continuity of the whole process, or without passing through a stage where the two particles are placed on the same point, and thus this loop is also nontrivial. By induction it can be shown that none of the nonzero powers of the loops of Fig. 12 is trivial. Therefore $\pi_1(F) = Z$, which is similar to the case of two points on a plane, and this result allows for *fractional* statistics for a set of three totally ordered points.

One might be tempted to believe that each triad of totally ordered points contributes a factor of Z to $\pi_1(F)$ when F contains four or more points; these factors may not be distinct, however, and may not even be Z, as one can show for a set of four totally ordered points. When looking at Fig. 15 one can see, for example, that F contains four distinct triads of

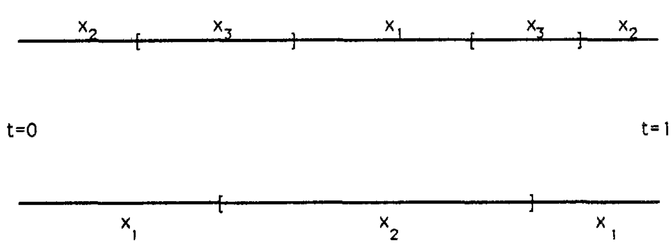

Fig. 13.

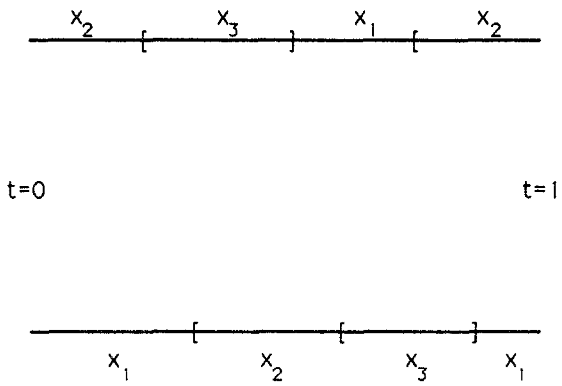

Fig. 14.

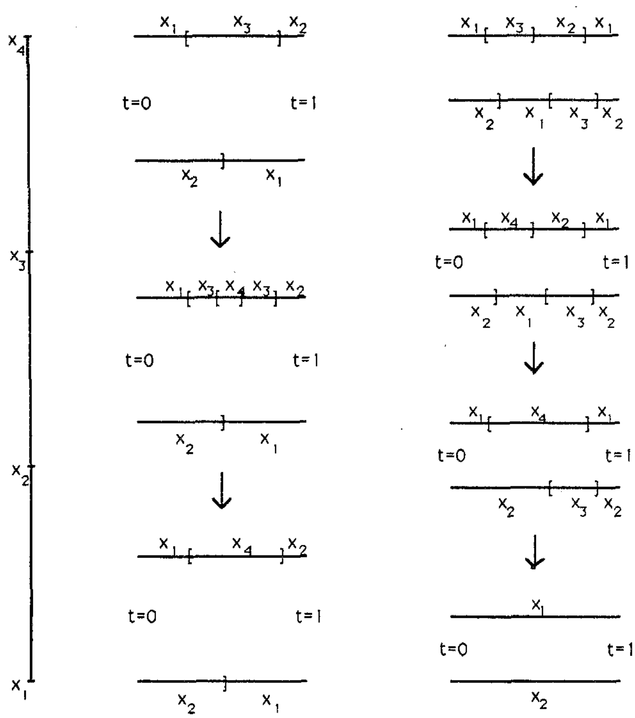

Fig. 15.

totally ordered points; on the same figure, however, we show that the exchange that uses the triad (x_1, x_2, x_3) is homotopic to the one that uses the triad (x_1, x_2, x_4). We also show that their squares are trivial, and thus $\pi_1(Q) = Z_2$, which means that only *Bose* and *Fermi* statistics are possible.

One does not need, on the other side, to have three at least totally ordered points in order to get a nontrivial fundamental group. Let $F = \{x_1, x_2, x_3, x_4\}$ and let its partial ordering be the one shown in Fig. 2. In Fig. 16 we show a nontrivial exchange, and in Fig. 17 its square, which is also nontrivial, since the trajectories of both particles are nontrivial. Similarly all powers of this exchange are nontrivial, and thus $\pi_1(F) = Z$, which allows for fractional statistics, even though F contains no totally ordered triads.

By studying other such examples one may see the following pattern emerging. One begins with the Hasse diagram, where once more diagonals do not "touch" each other. Then one looks for totally ordered n-ads $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$, and adds all possible lines linking x_{i_a} to x_{i_b} that do not already appear on the Hasse diagram in such a way that no two such links are collinear, no three are coplanar, etc. Finally, one "fills" the interiors of all triangles whose vertices are totally ordered triads. The possibilities for statistics on F are the same with the ones on the space formed the way

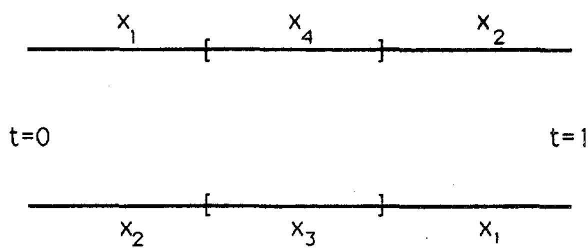

Fig. 16.

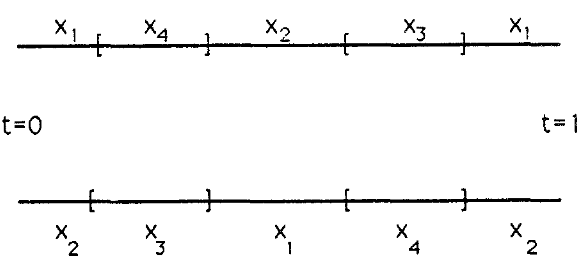

Fig. 17.

described above, where now instead of the topology derived from the partial ordering we use the usual one.

It is interesting to notice that if F contains no totally ordered tetrads and no triads have two points in common, one does not need to "fill" the triangles arising from the totally ordered triads, and thus the problem is reduced to the study of one-dimensional systems (not necessarily manifolds). Statistics on such networks have already been studied in the literature (Balachandran and Ercolessi, 1991).

If one is interested in the situation of three or more identical particles, the situation is even more involved. In general, for a system of n identical particles exchanges are always possible if F contains at least $n + 1$ totally ordered points, but may not be possible otherwise. For the simple case where F is totally ordered containing m points ($m > n$), the fundamental group can be shown to be equal to the *braid* group B_n for $m = n + 1$, and the *permutation* group S_n if $m > n + 1$. It should be noted that these are also the fundamental groups of the configuration spaces of n points on R^2 and R^k, where $k > 2$, respectively.

5. CONCLUSION

We have discussed some of the topological properties of finite partially ordered sets. It would be interesting if we could relate these properties to the ones of appropriate continuous spaces. It does not appear, for example, to be a mere coincidence that the statistics possible on a totally ordered set is exactly the same as that on R^2 or R^k.

When there is only one particle present, and thus the configuration space is identical to the physical one, Sorkin has argued that one may substitute a continuous space by a family of appropriate finite sets of open coverings, and get the same topological properties. For the problem of statistics, however, when one needs the presence of two or more identical particles, such a substitution cannot be taken for granted. One should notice, for example, that while two particles are forbidden to be located at the same point of a finite space, they are allowed to be inside an area of the continuous space which corresponds to a single point of the finite one.

We thus leave this question unresolved, and hope to return to it in the future.

ACKNOWLEDGMENTS

The work of the author was supported by a Sir Charles Clore fellowship of the Feinberg Graduate School at the Weizmann Institute of Science and by the BSF grant 5360/1. The author would like to thank

Miriam Leurer for her comments on the manuscript, and Rafael Sorkin and Adam Schwimmer for interesting discussions on the subject.

REFERENCES

Balachandran, A. P. (1991). Quantum statistics, in *Classical Topology and Quantum States*, A. P. Balachandran, G. Marmo, B. S. Skagerstam, and A. Stern, eds., World Scientific, Singapore.

Balachandran, A. P., and Ercolessi, E. (1991). Syracuse University preprint SU-4228-488.

Sorkin, R. D. (1986). Introduction to topological geons, in *Topological Properties and Global Structure of Spacetime*, P. G. Bergmann and V. de Sabbata, eds., Plenum Press, New York, and references therein.

Sorkin, R. D. (1991). *International Journal of Theoretical Physics*, **30**, 923.

Stong, R. E. (1966). *Transactions of the American Mathematical Society*, **123**, 325.